数理解析研究所講究録
第 1874 巻 2014 年 12-21

The explicit upper bound of the multiple integral of $S(t)$ on the Riemann Hypothesis

名古屋大学 多元数理科学研究科 若狭尊裕
Takahiro Wakasa

Graduate School of Mathematics, Nagoya University

Abstract

We prove explicit upper bounds of the function $S_m(T)$, defined by the repeated integration of the argument of the Riemann zeta-function. The explicit upper bound of $S(T)$ and $S_1(T)$ have already been obtained by A. Fujii. Our result is a generalization of Fujii's results.

1 Introduction

We consider the argument of the Riemann zeta function $\zeta(s)$, where $s=\sigma+ti$ is a complex variable, on the critical line $\sigma=\frac{1}{2}$.

We shall give some explicit bounds on $S_m(T)$ defined below under the Riemann hypothesis.

We introduce the functions $S(t)$ and $S_1(t)$. When $T\neq\gamma$ (γ is not the ordinate of any zero of $\zeta(s)$), we define

$$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+Ti\right).$$

This is obtained by continuous variation along the straight lines connecting 2, $2+Ti$, and $\frac{1}{2}+Ti$, starting with the value zero. When $T=\gamma$, we define

$$S(T)=\frac{1}{2}\{S(T+0)+S(T-0)\}.$$

Next, we define $S_1(T)$ by

$$S_1(T)=\int_0^T S(t)dt+C, \quad \left(C=\frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma)|d\sigma : constant\right).$$

It is a classical results of von Mangoldt (cf. chapter 9 of Titchmarsh [7]) that there exists a number $T_0>0$ such that for $T>T_0$ we have

$$S(T)=O(\log T), \quad S_1(T)=O(\log T).$$

Further, it is a classical result of Littlewood [8] that under the Riemann Hypothesis we have

$$S(T)=O\left(\frac{\log T}{\log\log T}\right), \quad S_1(T)=O\left(\frac{\log T}{(\log\log T)^2}\right).$$

For explicit upper bounds of $|S(T)|$ and $|S_1(T)|$, Karatsuba and Korolev (cf. Theorem 1 and Theorem 2 on [9]) have shown that

$$|S(T)|<8\log T, \quad |S_1(T)|<1.2\log T$$

for $T > T_0$. Also, under the Riemann Hypothesis, it was shown that

$$|S(T)| \le 0.83 \frac{\log T}{\log\log T}, \quad |S_1(T)| \le 0.51 \frac{\log T}{(\log\log T)^2}$$

for $T > T_0$ by Fujii.

Next, we introduce the functions $S_2(T)$, $S_3(T)$, $\cdots$. And the non-trivial zeros of $\zeta(s)$ we denote by $\rho = \beta + \gamma i$. When $T \neq \gamma$, we put

$$S_0(T) = S(T), \quad S_m(T) = \int_0^T S_{m-1}(t)dt + C_m$$

for any integer $m \ge 1$, where C_m's are the constants which are defined by, for any integer $k \ge 1$,

$$C_{2k-1} = \frac{1}{\pi}(-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{(2k-1)-times} \log|\zeta(\sigma)|(d\sigma)^{2k-1},$$

and

$$C_{2k} = (-1)^{k-1} \underbrace{\int_{\frac{1}{2}}^{\infty} \int_{\sigma}^{\infty} \cdots \int_{\sigma}^{\infty}}_{2k-times} (d\sigma)^{2k} = \frac{(-1)^{k-1}}{(2k)!2^{2k}}.$$

When $T = \gamma$, we put

$$S_m(T) = \frac{1}{2}\{S_m(T+0) + S_m(T-0)\}.$$

Concerning $S_m(T)$ for $m \ge 2$, Littlewood [8] have shown under the Riemann Hypothesis that

$$S_m(T) = O\left(\frac{\log T}{(\log\log T)^{m+1}}\right).$$

Theorem 1.
Under the Riemann Hypothesis for any integer $m \ge 1$, if m is odd,

$$\begin{aligned}|S_m(t)| \le \frac{\log t}{(\log\log t)^{m+1}} \cdot \frac{1}{2\pi m!} \Bigg\{ & \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1}e^2}\right) \\ & + \frac{1}{m+1} \cdot \frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{1}{m(m+1)} \cdot \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \Bigg\} \\ & + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).\end{aligned}$$

If m is even,

$$\begin{aligned}|S_m(t)| \le \frac{\log t}{(\log\log t)^{m+1}} \cdot \frac{1}{2\pi m!} \Bigg\{ & \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \sum_{j=0}^{m} \frac{m!}{(m-j)!} \left(\frac{1}{e} + \frac{1}{2^{j+1}e^2}\right) \\ & + \frac{1}{m+1} \cdot \frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{\pi}{2} \cdot \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} \Bigg\} + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).\end{aligned}$$

This result is a generalization of the known explicit upper bounds for $S(T)$ and $S_1(T)$. It is to be stressed that the argument when the number of integration is odd is different from that when the number of integration is even.

The basic policy of the proof of this result is based on A. Fujii [1]. In the case when m is odd, we can directly generalize the proof of A. Fujii [1]. In the case when m is even, it is an extension of the method of A. Fujii [2].

To prove this result, we introduce some more notations. First, we define the function $I_m(T)$ as follows. When $T \neq \gamma$, we put for any integer $k \geq 1$

$$I_{2k-1}(T) = \frac{1}{\pi}(-1)^{k-1}\Re\Big\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{(2k-1)-times}\log\zeta(\sigma+Ti)(d\sigma)^{2k-1}\Big\}$$

and

$$I_{2k}(T) = \frac{1}{\pi}(-1)^{k}\Im\Big\{\underbrace{\int_{\frac{1}{2}}^{\infty}\int_{\sigma}^{\infty}\cdots\int_{\sigma}^{\infty}}_{2k-times}\log\zeta(\sigma+Ti)(d\sigma)^{2k}\Big\}.$$

When $T = \gamma$, we put for $m \geq 1$

$$I_m(T) = \frac{1}{2}\{I_m(T+0) + I_m(T-0)\}.$$

Then, $I_m(T)$ can be expressed as a single integral of the following form (cf. Lemma 2 in Fujii [3]): for any integer $m \geq 1$

$$I_m(T) = -\frac{1}{\pi}\Im\left\{\frac{i^m}{m!}\int_{\frac{1}{2}}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\frac{\zeta'}{\zeta}(\sigma+Ti)d\sigma\right\}.$$

From this expression, it is known under the Riemann Hypothesis that $S_m(T) = I_m(T)$ by Lemma 2 in Fujii [4].

Therefore, we should estimate $I_m(T)$.

2 Some lemmas

Let $s = \sigma + ti$. We suppose that $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let X be a positive number satisfying $4 \leq X \leq t^2$. Also, we put

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log X}, \quad \Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq X, \\ \Lambda(n)\frac{\log\frac{X^2}{n}}{\log X} & \text{for } X \leq n \leq X^2, \end{cases}$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.
Let $t \geq 2$, $X > 0$ such that $4 \leq X \leq t^2$. For $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log X}$,

$$\begin{aligned}\frac{\zeta'}{\zeta}(\sigma+ti) = &-\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+ti}} - \frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right) \\ &+ \frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t + O\left(X^{\frac{1}{2}-\sigma}\right),\end{aligned}$$

where $|\omega| \leq 1, -1 \leq \omega' \leq 1$.

This has been proved in Fujii [1].

Lemma 2. (cf. 2.12.7 of Titchmarsh[7])

$$\frac{\zeta'}{\zeta}(s) = \log 2\pi - 1 - \frac{E}{2} - \frac{1}{s-1} - \frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$
$$= \log 2\pi - 1 - \frac{E}{2} - \frac{1}{s-1} - \frac{1}{2}\log\left(\frac{s}{2}+1\right) + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + O\left(\frac{1}{|s|}\right)$$

where E is the Euler constant and ρ runs through zeros of $\zeta(s)$.

Lemma 3. (Lemma 1 of Selberg [6])
For $X > 1$, $s \neq 1$, $s \neq -2q$ $(q = 1, 2, 3, \cdots)$, $s \neq \rho$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<X^2}\frac{\Lambda_X(n)}{n^s} + \frac{X^{2(1-s)}-X^{1-s}}{(1-s)^2\log X} + \frac{1}{\log X}\sum_{q=1}^{\infty}\frac{X^{-2q-s}-X^{-2(2q+s)}}{(2q+s)^2} + \frac{1}{\log X}\sum_{\rho}\frac{X^{\rho-s}-X^{2(\rho-s)}}{(s-\rho)^2}.$$

By Lemma 2, we have

$$\Re\frac{\zeta'}{\zeta}(\sigma_1+ti) = -\frac{1}{2}\log t + \sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2} + O(1). \tag{1}$$

Since for $\sigma_1 \leq \sigma$

$$\frac{1}{\log X}\left|\sum_{\rho}\frac{X^{\rho-s}-X^{2(\rho-s)}}{(s-\rho)^2}\right| \leq \left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2},$$

we have

$$\frac{1}{\log X}\sum_{\rho}\frac{X^{\rho-s}-X^{2(\rho-s)}}{(s-\rho)^2} = \left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}\cdot\omega\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2},$$

where $|\omega| \leq 1$. Since for $\sigma \geq \frac{1}{2}$ and $X \leq t^2$

$$\left|\frac{X^{2(1-s)}-X^{1-s}}{(1-s)^2\log X}\right| \ll \frac{X^{2(1-\sigma)}}{t^2\log X} \leq \frac{X^{\frac{1}{2}-\sigma}}{\log X},$$

we have for $\sigma_1 \leq \sigma$

$$\frac{\zeta'}{\zeta}(\sigma+ti) = -\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+ti}} + O\left(\frac{X^{\frac{1}{2}-\sigma}}{\log X}\right) + \left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}$$

by Lemma 3. Especially,

$$\Re\frac{\zeta'}{\zeta}(\sigma_1+ti) = \Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right) + O\left(\frac{1}{\log X}\right) + \left(1+\frac{1}{e}\right)\frac{1}{e}\omega'\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}, \tag{2}$$

where $-1 \leq \omega' \leq 1$.

Hence by (1) and (2), we get

$$\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2} = \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t + O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right). \tag{3}$$

This relation will be used in the following proof of Theorem 1.

3 Proof of Theorem 1 in the case when m is odd

If m is odd, we have

$$\begin{aligned} I_m(t) &= \frac{i^{m+1}}{\pi m!}\Im\left\{i\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\frac{\zeta'}{\zeta}(\sigma+ti)d\sigma+\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{m+1}\cdot\frac{\zeta'}{\zeta}(\sigma_1+ti)\right.\right. \\ &\quad \left.\left.-\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\left\{\frac{\zeta'}{\zeta}(\sigma_1+ti)-\frac{\zeta'}{\zeta}(\sigma+ti)\right\}d\sigma\right\}\right\} \\ &= \frac{i^{m+1}}{\pi m!}\Im\left\{i(J_1+J_2+J_3)\right\}, \end{aligned}$$

say.

First, we estimate J_1. By Lemma 1,

$$\begin{aligned} J_1 &= \int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left\{-\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+ti}}-\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right)\right. \\ &\quad \left.+\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(X^{\frac{1}{2}-\sigma}\right)\right\}d\sigma \\ &= -\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+ti}}d\sigma+\eta_1(t), \\ &= -\sum_{j=0}^{m}\left(\frac{m!}{(m-j)!}\left(\sigma_1-\frac{1}{2}\right)^{m-j}\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}(\log n)^{j+1}}\right)+\eta_1(t), \end{aligned}$$

say. And we have

$$\begin{aligned} |\eta_1(t)| &= \left|\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}d\sigma\right|\cdot\left|-\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right)+\frac{1}{2}\log t\right| \\ &\quad +O\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m X^{\frac{1}{2}-\sigma}d\sigma\right\} \\ &\le \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right) \\ &\quad +O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right) \\ &= \eta_2(t)+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right), \end{aligned}$$

say, since by partial integration

$$\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left(1+X^{\frac{1}{2}-\sigma}\right)X^{\frac{1}{2}-\sigma}d\sigma=\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right).$$

Next, applying Lemma 1 to J_2, we get

$$\begin{aligned} J_2 &= \frac{1}{(m+1)(\log X)^{m+1}}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right\} \\ &= \eta_3(t)+O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right\}, \end{aligned}$$

say.

Next, we estimate J_3. By Lemma 2, we have

$$\Im(iJ_3) = -\sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \cdot \int_{\frac{1}{2}}^{\sigma_1} \left(\sigma - \frac{1}{2}\right)^m \frac{(\sigma_1 - \sigma)\left\{(t-\gamma)^2 - \left(\sigma_1 - \frac{1}{2}\right)\left(\sigma - \frac{1}{2}\right)\right\}}{\left(\sigma - \frac{1}{2}\right)^2 + (t-\gamma)^2} d\sigma$$

$$+ O\left(\frac{1}{t(\log X)^{m+1}}\right)$$

$$= -\sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \cdot K(\gamma) + O\left(\frac{1}{(\log X)^{m+1}}\right),$$

say, where γ is the imaginary part of $\rho = \beta + \gamma i$.

If $t = \gamma$,

$$K(\gamma) = -\frac{1}{m(m+1)}\left(\sigma_1 - \frac{1}{2}\right)^{m+2}.$$

If $t \neq \gamma$, by putting $\sigma - \frac{1}{2} = v$, $\sigma_1 - \frac{1}{2} = \frac{1}{\log X} = \Delta$ and $|t - \gamma| = B$, we get

$$K(\gamma) = \int_0^{\Delta} v^m \frac{(\Delta - v)(B^2 - \Delta v)}{v^2 + B^2} dv = \frac{\Delta^{m+2}}{m+1} - \frac{(B^2+\Delta^2)\Delta^m}{m} + \int_0^{\Delta} \frac{(B^2+\Delta^2)v^{m-1}}{\left(\frac{v}{B}\right)^2 + 1} dv.$$

Putting $\frac{v}{B} = u$, we have

$$K(\gamma) = \frac{\Delta^{m+2}}{m+1} - \frac{(B^2+\Delta^2)\Delta^m}{m} + (B^2+\Delta^2)\int_0^{\frac{\Delta}{B}} \frac{(uB)^{m-1}B}{1+u^2} du$$

$$= \Delta^{m+2}\left\{\frac{1}{m+1} - \frac{B^2}{m\Delta^2} - \frac{1}{m} + \left(\frac{B^{m+2}}{\Delta^{m+2}} + \frac{B^m}{\Delta^m}\right) i^{m+1}\left\{\sum_{j=1}^{\frac{m-1}{2}} \frac{(-1)^{j-1}}{2j-1}\left(\frac{\Delta}{B}\right)^{2j-1} - \arctan\left(\frac{\Delta}{B}\right)\right\}\right\}.$$

Putting $y = \frac{\Delta}{B}$, we get

$$K(\gamma) = \Delta^{m+2}\left(g(y) - \frac{1}{m(m+1)}\right), \tag{4}$$

where

$$g(y) = \left\{-i^{m+1}\left(\frac{1}{y^{m+2}} + \frac{1}{y^m}\right)\arctan y - \frac{1}{my^2} + i^{m+1}\left(\frac{1}{y^{m+2}} + \frac{1}{y^m}\right)\sum_{j=1}^{\frac{m-1}{2}} \frac{(-1)^{j-1}}{2j-1} y^{2j-1}\right\}.$$

When y tends to 0, $g(y)$ is convergent to $\frac{2}{m(m+2)}$. When y tends to infinity, $g(y)$ tends to 0. Hence for $y > 0$, we get $g'(y) < 0$, so that

$$-\frac{1}{m(m+1)} \leq g(y) - \frac{1}{m(m+1)} \leq \frac{1}{(m+1)(m+2)}. \tag{5}$$

Therefore by (4) and (5), we obtain

$$-\frac{1}{m(m+1)}\left(\sigma_1 - \frac{1}{2}\right)^{m+2} \leq K(\gamma) \leq \frac{1}{(m+1)(m+2)}\left(\sigma_1 - \frac{1}{2}\right)^{m+2}.$$

Hence

$$-\sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} K(\gamma) \leq \frac{\left(\sigma_1 - \frac{1}{2}\right)^{m+2}}{m(m+1)} \sum_{\gamma} \frac{1}{\left(\sigma_1 - \frac{1}{2}\right)^2 + (t-\gamma)^2} \tag{6}$$

and

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)\geq-\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{(m+1)(m+2)}\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}. \tag{7}$$

By (3), (6) and (7), we have

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)\leq\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{m(m+1)}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\right\}$$

and

$$-\sum_{\gamma}\frac{1}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}K(\gamma)\geq-\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+1}}{(m+1)(m+2)}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\right\}.$$

Hence

$$\begin{aligned}|i^{m+1}\Im(iJ_3)|&\leq\frac{1}{m(m+1)}\cdot\frac{1}{(\log X)^{m+1}}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\\&=\eta_5(t)+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}I_m(t)=\frac{1}{\pi m!}\Bigg\{&-i^{m+1}\sum_{j=0}^{m}\left(\frac{m!}{(m-j)!}\left(\sigma_1-\frac{1}{2}\right)^{m-j}\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}(\log n)^{j+1}}\right)\\&+O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\Bigg\}+\frac{1}{\pi m!}\cdot\Xi(t),\end{aligned} \tag{8}$$

where $\Xi(t)$ satisfies the following inequalities.

$$\begin{aligned}|\Xi(t)|\leq&\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right)\\&+\frac{1}{m+1}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\\&+\frac{1}{m(m+1)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}.\end{aligned}$$

In (8), we have

$$\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\leq\sum_{n<X}\frac{\Lambda(n)}{n^{\frac{1}{2}}}+\sum_{X\leq n\leq X^2}\frac{\Lambda(n)\log\frac{X^2}{n}}{n^{\frac{1}{2}}}\cdot\frac{1}{\log X}\ll\frac{X}{\log X}, \tag{9}$$

$$\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}(\log n)^{j+1}}\right|\leq\sum_{n<X}\frac{\Lambda(n)}{n^{\frac{1}{2}}(\log n)^{j+1}}+\sum_{X\leq n\leq X^2}\frac{\Lambda(n)\log\frac{X^2}{n}}{n^{\frac{1}{2}}(\log n)^{j+1}}\cdot\frac{1}{\log X}\ll\frac{X}{(\log X)^{j+2}}. \tag{10}$$

We estimate that the first term and the second term on the right-hand side of (8) is $\ll\frac{X}{(\log X)^{m+2}}$.

Therefore, taking $X = \log t$, we obtain

$$
\begin{aligned}
|I_m(t)| &= \frac{1}{\pi m!}\Xi(t) + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right) \\
&= \frac{\log t}{(\log\log t)^{m+1}} \cdot \frac{1}{2\pi m!}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right. \\
&\quad \left. + \frac{1}{m+1}\cdot\frac{\frac{1}{e}\left(1+\frac{1}{e}\right)}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)} + \frac{1}{m(m+1)}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\right\} + O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).
\end{aligned}
$$

This is the first part of the result.

4 Proof of Theorem 1 in the case when m is even

If m is even, we get similarly

$$
\begin{aligned}
I_m(t) &= \frac{-i^m}{\pi m!}\Im\left\{\left\{\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\frac{\zeta'}{\zeta}(\sigma+ti)d\sigma + \frac{(\sigma_1-\frac{1}{2})^{m+1}}{m+1}\cdot\frac{\zeta'}{\zeta}(\sigma_1+ti)\right.\right. \\
&\quad \left.\left. -\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\left\{\frac{\zeta'}{\zeta}(\sigma_1+ti)-\frac{\zeta'}{\zeta}(\sigma+ti)\right\}d\sigma\right\}\right\} \\
&= \frac{-i^m}{\pi m!}\Im\left\{(J_1+J_2+J_3)\right\},
\end{aligned}
$$

say. By Lemma 1 and (9), we have

$$
\begin{aligned}
J_1 &= -\int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma+ti}}d\sigma + \int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m O\left(X^{\frac{1}{2}-\sigma}\right)d\sigma \\
&\quad + \int_{\sigma_1}^{\infty}\left(\sigma-\frac{1}{2}\right)^m\left\{-\frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right) + \frac{\left(1+X^{\frac{1}{2}-\sigma}\right)\omega X^{\frac{1}{2}-\sigma}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t\right\}d\sigma \\
&= \sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\sigma_1+\frac{1}{2}\right)^{m-j}\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}(\log n)^{j+1}} + O\left(\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right) + \eta_1'(t) \\
&\ll \frac{X}{(\log X)^{m+2}} + \eta_1'(t),
\end{aligned}
$$

say, and

$$
\begin{aligned}
J_2 &= \frac{1}{(m+1)(\log X)^{m+1}}\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}} - \frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\Re\left(\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right)\right. \\
&\quad \left. + \frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t + O\left(X^{\frac{1}{2}-\sigma_1}\right)\right\} \\
&= \frac{1}{(m+1)(\log X)^{m+1}}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}\omega}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t + O\left\{\frac{1}{(\log X)^{m+1}}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right\} \\
&\ll \eta_3'(t) + \frac{X}{(\log X)^{m+2}},
\end{aligned}
$$

say. As well as $\eta_1(t)$, we have

$$
|\eta_1'(t)| \le \frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{1}{2}\log t\cdot\frac{1}{(\log X)^{m+1}}\left(\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right).
$$

Finally, we estimate J_3. By Stirling's formula, we get

$$\left|\frac{\Gamma'}{\Gamma}\left(\frac{\sigma_1+ti}{2}+1\right)\right| = \left|\frac{i}{2}\log\frac{ti}{2}+\left(\frac{\sigma_1+ti+1}{2}\right)\frac{1}{t}-\frac{i}{2}+O\left(\frac{1}{t}\right)\right| \leq \frac{1}{2}\log t+O\left(\frac{1}{t}\right). \tag{11}$$

Also $\left|\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+ti}{2}+1\right)\right|$ is estimated similarly.

Hence by (11) and Lemma 2, we have

$$\left|\Im\left\{\frac{\zeta'}{\zeta}(\sigma_1+ti)-\frac{\zeta'}{\zeta}(\sigma+ti)\right\}\right| \leq \sum_{\gamma}\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}+O\left(\frac{1}{t}\right).$$

Therefore,

$$\begin{aligned}|\Im(J_3)| \leq &\left|\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\sum_{\gamma}\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}d\sigma\right| \\ &+\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\cdot O\left(\frac{1}{t}\right)d\sigma.\end{aligned}$$

If $t=\gamma$, the first term of the right-hand side of above inequality is 0. If $t\neq\gamma$, since $\sigma<\sigma_1$, we have

$$\begin{aligned}&\left|\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\left\{\sum_{\gamma}\frac{(t-\gamma)\left\{\left(\sigma-\frac{1}{2}\right)^2-\left(\sigma_1-\frac{1}{2}\right)^2\right\}}{\left\{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}\left\{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2\right\}}\right\}d\sigma\right| \\ &<\sum_{\gamma}\frac{\left(\sigma_1-\frac{1}{2}\right)^{m+2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}\int_{\frac{1}{2}}^{\infty}\frac{|t-\gamma|}{\left(\sigma-\frac{1}{2}\right)^2+(t-\gamma)^2}d\sigma \leq \frac{\pi}{2}\left(\sigma_1-\frac{1}{2}\right)^{m+1}\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{\left(\sigma_1-\frac{1}{2}\right)^2+(t-\gamma)^2}.\end{aligned}$$

Applying (3) and (9), and taking $X=\log t$ lastly, the right-hand side of above inequality is

$$\begin{aligned}&\leq\frac{\pi}{2}\left(\sigma_1-\frac{1}{2}\right)^{m+1}\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)\omega'}\cdot\frac{1}{2}\log t+O\left(\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+ti}}\right|\right)\right\} \\ &\leq\frac{\pi}{4}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{\log t}{(\log\log t)^{m+1}}+O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).\end{aligned} \tag{12}$$

Also,

$$\int_{\frac{1}{2}}^{\sigma_1}\left(\sigma-\frac{1}{2}\right)^m\cdot O\left(\frac{1}{t}\right)d\sigma=O\left(\frac{1}{t(\log X)^{m+1}}\right). \tag{13}$$

By (12) and (13),

$$|\Im(J_3)|\leq\frac{\pi}{4}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\cdot\frac{\log t}{(\log\log t)^{m+1}}+O\left(\frac{1}{t(\log\log t)^{m+1}}\right)+O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).$$

Therefore, we obtain

$$\begin{aligned}|S_m(t)|\leq\frac{1}{2\pi m!}\cdot\frac{\log t}{(\log\log t)^{m+1}}&\left\{\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\sum_{j=0}^{m}\frac{m!}{(m-j)!}\left(\frac{1}{e}+\frac{1}{2^{j+1}e^2}\right)\right. \\ &\left.+\frac{1}{m+1}\cdot\frac{\left(1+\frac{1}{e}\right)\frac{1}{e}}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}+\frac{\pi}{2}\cdot\frac{1}{1-\frac{1}{e}\left(1+\frac{1}{e}\right)}\right\}+O\left(\frac{\log t}{(\log\log t)^{m+2}}\right).\end{aligned}$$

□

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